Math 261B Tues. Nov. 17

$U_q(sl_2)$ $\qquad B \subseteq SL_2 \qquad k = \mathbb{Q}(q)$

Recall $\quad \mathcal{O}_q(B) = k\langle t^{\pm 1}, x\rangle / (txt^{-1} - qx)$

$$\Delta t = t \otimes t$$

$$\Delta x = x \otimes t^{-1} + t \otimes t$$

$$U_v(b) = k\langle K^{\pm 1/2}, E\rangle / (K^{1/2} E K^{-1/2} - qE)$$

$$\Delta K^{1/2} = K^{1/2} \otimes K^{1/2}$$

$$\Delta E = E \otimes K^{-1/2} + K^{1/2} \otimes E$$

New $E = K^{1/2} E_{old}$

$$KEK^{-1} = q^2 E$$

$$\Delta K = K \otimes K$$

$$\Delta E = E \otimes 1 + K \otimes E$$

$$\left[= (K^{1/2} \otimes K^{1/2}) \cdot (E_{old} \otimes K^{-1/2} + K^{1/2} \otimes E_{old}) \right]$$

Def $U_q(sl_2) = \mathbb{A}\langle E, F, K\rangle \Big/ \left(KEK^{-1} - q^2E,\ KFK^{-1} - q^{-2}F,\ [E,F] - \frac{K-K^{-1}}{q-q^{-1}} \right)$

Note $K \mapsto q^m$ $\quad \frac{K-K^{-1}}{q-q^{-1}} \mapsto \underbrace{\frac{q^m-q^{-m}}{q-q^{-1}}}_{(m)_q} = q^{m-1} + q^{m-3} + \cdots + q^{1-m}$ ← m terms

$(m)_q$ q-analog of m. $\quad (-m)_q = -(m)_q$

$\{ F^r K^m E^s \mid r,s \in \mathbb{N}\ \ m \in \mathbb{Z} \}$ is a basis of $U_q(sl_2)$

$\Delta K = K \otimes K \qquad \Delta E = E \otimes 1 + K \otimes E \qquad \Delta F = F \otimes K^{-1} + 1 \otimes F$

Verifying that this works

Coassociativity

$$\Delta^{(n)} K = K \otimes K \otimes \cdots \otimes K$$

$$\Delta^{(n)} E = \sum K \otimes K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1$$

$$\Delta^{(n)} F = \sum 1 \otimes \cdots \otimes 1 \otimes F \otimes K^{-1} \otimes \cdots \otimes K^{-1}$$

Consistent with relations — specifically $[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$

$$[\Delta E, \Delta F] \overset{?}{=} \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}$$

$$\|$$

$$[E\otimes 1 + K\otimes E,\ F\otimes K^{-1} + 1\otimes F]$$

$$\|$$

$$[E\otimes 1, F\otimes K^{-1}] + [K\otimes E, 1\otimes F]$$

$$\|$$

$$[E,F]\otimes K^{-1} + K\otimes [E,F]$$

$$\|$$

$$\frac{K-K^{-1}}{q-q^{-1}}\otimes K^{-1} + K\otimes \frac{K-K^{-1}}{q-q^{-1}}$$

$$\|$$

$$\frac{\cancel{K\otimes K^{-1}} - K^{-1}\otimes K^{-1} + K\otimes K - \cancel{K\otimes K^{-1}}}{q-q^{-1}} = \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}$$

$$[E\otimes 1, 1\otimes F] = 0$$

$$E\otimes 1 \cdot 1\otimes F = E\otimes F$$
$$= 1\otimes F \cdot 1\otimes E$$

$$[K\otimes E, F\otimes K^{-1}] = 0$$

$$K\otimes E \cdot F\otimes K^{-1} = KF\otimes EK^{-1}$$
$$= q^{-2} FK \otimes EK^{-1}$$

$$F\otimes K^{-1} \cdot K\otimes E = FK\otimes K^{-1}E$$
$$= q^{-2} FK\otimes EK^{-1}$$

Representations of $U_q(sl_2)$ ("standard")

$\wedge$
V

want $\mathcal{O}_q(T)$ comodules $\Rightarrow$ $V = \bigoplus_{m\in\mathbb{Z}} V_m$

$\mathcal{O}_q(T) = A[t^{\pm 1}]$, $\Delta t = t\otimes t$

$Kv = q^m v$ for $v\in V_m$.

Non-standard modules exist: example

$V = A$, $E, F \to (0)$ $K \mapsto (-1)$

Examples V^m = q-analog of usual sl_2 module $V^m = k[x,y]_m$

$$\frac{y^m}{m!} \quad \frac{y^{m-1}x}{(m-1)!\,1!} \quad \cdots \quad \frac{x^{m-1}y}{(m-1)!} \quad x^m/m!$$

$$\begin{array}{ccccccc} \circlearrowleft_{-m} & & & & \circlearrowleft_{m-2} & & \circlearrowleft_{m} \\ E: & \xrightarrow{1} & \xrightarrow{2} & \cdots & & \xrightarrow{m} & \\ F: & \xleftarrow[m]{} & \xleftarrow[m-1]{} & & & \xleftarrow[1]{} & \end{array}$$

Quantum V^m

$$\begin{array}{ccccccccccc} E & & \xrightarrow{(1)_q} & & \xrightarrow{(2)_q} & & & & \xrightarrow{(m)_q} & \\ & \circ & & \circ & & \circ & \cdots & \circ & & \circ \\ F & & \xleftarrow[(m)_q]{} & & \xleftarrow[(m-1)_q]{} & & & & \xleftarrow[(1)_q]{} & \\ K & q^{-m} & & q^{2-m} & & q^{4-m} & & q^{m-2} & & q^m \end{array}$$

$$E^{(k)} = E^k/(k)_q$$

Check relations: $KE = q^2 EK$

If $Kv = q^r v$

$$KEv = q^2 EKv$$

$$= q^2 E q^r v = q^{r+2} Ev$$

$$K(Ev) = q^{r+2} Ev$$

$$\begin{array}{ccc} & E & \\ v & \longmapsto & Ev \\ K \circlearrowleft q^r & & K \circlearrowleft q^{r+2} \end{array}$$

$KF = q^{-2}FK \implies F$ lowers weight by 2

$[E,F] \overset{?}{=} \dfrac{K-K^{-1}}{q-q^{-1}}$

$$\begin{tikzcd}
 & v \arrow[d] & \\
\circ \arrow[r, bend left, "(k)_q"] & \circ \arrow[l, bend left, "(m-k+1)_q"] \arrow[r, bend left, "(k+1)_q"] \arrow[loop below, "q^{2k-m}"] & \circ \arrow[l, bend left, "(m-k)_q"]
\end{tikzcd}$$

$$[E,F]\,v = \big((k)_q(m-k+1)_q - (k+1)_q(m-k)_q\big)\,v$$

$$\frac{K-K^{-1}}{q-q^{-1}}\,v = (2k-m)_q\,v$$

$$(k)_q(m-k+1)_q - (k+1)_q(m-k)_q \overset{?}{=} (2k-m)_q$$

LHS

$$\frac{1}{(q-q^{-1})^2}\Big((q^k - q^{-k})(q^{m-k+1} - q^{-m+k-1}) - (q^{k+1} - q^{-k-1})(q^{m-k} - q^{k-m})\Big)$$

$$-q^{2k-m-1} \quad - q^{m+1-2k} \quad + q^{2k-m+1} \quad + q^{m-1-2k}$$

$$(q-q^{-1})\,q^{2k-m} \qquad -(q-q^{-1})\,q^{m-2k}$$

$$\frac{(q-q^{-1})(q^{2k-m} - q^{m-2k})}{(q-q^{-1})^2}$$

$$= (2k-m)_q$$

What does Δ do? It lets you tensor representations.

Say $\mathcal{U}$ is ~~a Hopf~~ an algebra over A (non-commutative)

V, W $\mathcal{U}$-modules. $V \underset{A}{\otimes} W$ is a $\mathcal{U} \underset{A}{\otimes} \mathcal{U}$ module

$\mathcal{U} \underset{A}{\otimes} \mathcal{U}'$ is algebra with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$

$\mathcal{U} \underset{A}{\otimes} 1 \cong \mathcal{U}$ $\quad 1 \underset{A}{\otimes} \mathcal{U}' \cong \mathcal{U}'$ commute

$\sum a_i \otimes b_i$

$$(a \otimes b) \cdot (v \otimes w) = av \otimes bw$$

To make $V \otimes_A W$ a $\mathcal{U}$ module, need an algebra homomorphism

$$\mathcal{U} \xrightarrow[\Delta]{} \mathcal{U} \underset{A}{\otimes} \mathcal{U} \curvearrowright V \otimes_A W$$

$(V_1 \underset{A}{\otimes} V_2) \underset{A}{\otimes} V_3 \cong V_1 \underset{A}{\otimes} (V_2 \underset{A}{\otimes} V_3)$ $\leftarrow$ get same $\mathcal{U}$ action

$\Updownarrow$

$(a \otimes b) \otimes c \longleftrightarrow a \otimes (b \otimes c)$ $\quad$ Δ is coassociative

$U \xrightarrow[\varepsilon]{} A$ makes A a U-module

$V \otimes_A A \cong V$

same U module

$\Longleftarrow$ ε is a counit.

$\mathrm{Hom}_A(V, A) = V^* \longleftarrow U^{op}$ acts

$V \xrightarrow{\varphi} V \xrightarrow{\psi} V$

$V \xleftarrow{\varphi^*} V^* \xleftarrow{\psi^*} V^*$

$(V \otimes W)^* \leftarrow V^* \otimes W^*$

U hom.

$U \xrightarrow[S]{} U^{op}$ makes V^* a U module

<u>Example</u>

V^1: $F \downarrow$ u_0 ↻ q, v_0 ↻ q^{-1}, E $\uparrow$, 1, K, $(1)_q$

$V^1 \otimes V^1$

$$\begin{tikzcd}
 & u\otimes u \arrow[dl, shift left, "1"] \arrow[dr, shift left, "q^{-1}"] & \\
u\otimes v \arrow[ur, shift left, "q"] \arrow[dr, shift left, "q"] & & v\otimes u \arrow[ul, shift left, "1"] \arrow[dl, shift left, "1"] \\
 & v\otimes v \arrow[ul, shift left, "1"] \arrow[ur, shift left, "q^{-1}"] &
\end{tikzcd}$$

$E \uparrow \quad \downarrow F$

↻ q^2

↻ 1 $\quad K$

↻ q^{-2}

$\Delta E = E \otimes 1 + K \otimes E$ $\qquad$ $\Delta F = F \otimes K^{-1} + 1 \otimes F$

$$u \otimes u \quad \circlearrowleft \; q^2$$

$$q + q^{-1} \quad (2)_q \uparrow E \qquad \downarrow (1)_q \; F$$

$$u \otimes v + q^{-1}\, v \otimes u \quad \circlearrowleft \; 1 \qquad K$$

$$E \uparrow \qquad \downarrow (2)_q$$

$$v \otimes v \quad \circlearrowleft \; q^{-2}$$

Submodule $\cong V^2$

$$0 \xleftarrow{E} u \otimes v - q\, v \otimes u \xrightarrow{F} 0 \qquad \circlearrowleft^{1} K$$

V^0

$$V^1 \otimes V^1 \cong V^2 \oplus V^0$$